

# Zero-mean Gaussian을 이용한 소구간 사다리꼴공식의 오차

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## 요 약

이 논문에서는 정적분의 수치계산 방법 중에 하나인 사다리꼴 공식의 평균오차를 zero mean-Gaussian을 이용하여 연구한다. 구간  $[0,1]$ 에  $n$ 개의 소구간을 잡고 계산의 단순화를 위하여 각 소구간의 길이가 같다고 하고 길이를  $h$ 라 하면,  $r \leq 2$ 일 때, 상수  $c_r$ 을 직접 계산하여 연속된 두 개의 소구간 위에서 단순 사다리꼴공식과 복합 사다리꼴공식 사이의 평균오차가  $O(h^{2r+3})$ 임을 보인다.

키워드 : 평균오차, 수치구적법, 사다리꼴공식

## An Error Bound of Trapezoidal Rule on Subintervals using Zero-mean Gaussian

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### ABSTRACT

In this paper, we study the average case error of the Trapezoidal rule using zero mean-Gaussian. Assume that we have  $n$  subintervals (for simplicity equal length) partitioning  $[0,1]$  and that each subinterval has the length  $h$ . Then, for  $r \leq 2$ , we show that the average error between simple Trapezoidal rule and the composite Trapezoidal rule on two consecutive subintervals is bounded by  $h^{2r+3}$  through direct computation of constants  $c_r$ .

Key Words : Average Case Error, Numerical Integration, Trapezoidal Rule

### 1. Introduction

Many numerical computations in science and engineering can only be solved approximately since the available information is partial. For instance, for problems defined on a space of functions, information about  $f$  is typically provided by a few function values,

$$N(f) = [f(x_1), f(x_2), \dots, f(x_n)].$$

Knowing  $N(f)$ , the solution is approximated by a numerical method.

The error between the true solution and the approximation depends on a problem setting. In the **worst case**

**setting**, the error of a numerical scheme is defined by its worst performance with respect to the given class of functions; see [1, 4] and [6]. In this paper, we concentrate on another setting, the **average case setting**. In this setting, we assume that the class  $F$  of input functions is equipped with a probability measure. Then the average case error of an algorithm is defined by its expectation, rather than by its worst case performance. The average case analysis is important and significant number of results have already been obtained (see, e.g., [6] and the references cited therein).

### 2. Definitions

It is well known that the average case setting requires the space of functions to be equipped with a probability measure. In this paper, we choose a probability measure  $\mu_r$  which is a variant of an  $r$ -fold Wiener measure  $\omega_r$ ; see [3, 5] and [6]. The probability measure  $\omega_r$  is a

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Gaussian measure with zero mean and correlation function given by

$$M_{\omega_r}(f(x)f(y)) = \int_F f(x)f(y) \omega_r(df) = \int_0^1 \frac{(x-t)_+^r}{r!} \frac{(y-t)_+^r}{r!} dt,$$

where  $(z-t)_+^r = [\max\{0, z-t\}]^r$ . Equivalently,  $f$  distributed according to  $\omega_r$  can be viewed as a Gaussian stochastic process with zero mean and autocorrelation given above. However, since  $\omega_r$  is concentrated on functions with boundary conditions  $f(0) = f'(0) = \dots = f^{(r)}(0) = 0$ , we choose to study a slightly modified measure  $\mu_r$  that preserves basic properties of  $\omega_r$ , yet does not require any boundary conditions. More precisely, we assume that a function  $f$ , as a stochastic process, is given by  $f(x) = f_1(x) + f_2(1-x)$ ,  $x \in [0, 1]$ , where  $f_1$  and  $f_2$  are independent and distributed according to  $\omega_r$ .

In this paper, as the class  $F$  of input functions, we choose the space  $F = C^r[0, 1]$  that is equipped with a probability measure  $\mu_r$  which is a variant of the  $r$ -fold **Wiener measure**. In order to define it, if we first recall basic properties of the classical  $r$ -fold **Wiener measure**  $\omega_r$ , then according to [2-5], the probability measure  $\mu_r$  defined on  $\sigma$ -field of the space  $C^r[0, 1]$  is zero mean Gaussian with the correlation function given by

$$M_{\mu_r}(f(x), f(y)) = \int_0^1 \frac{(x-t)_+^r (y-t)_+^r + (1-x-t)_+^r (1-y-t)_+^r}{r! r!} dt = \int_0^1 \frac{(x-t)_+^r (y-t)_+^r + (t-x)_+^r (t-y)_+^r}{r! r!} dt.$$

We study the problem of approximating an integral  $I(f) = \int_0^1 f(x) dx$  for  $f \in F = C^r[0, 1]$ , assuming that the class of integrands is equipped with the probability measure  $\mu_r$ .

Assume that we have  $n$  subintervals (not necessarily equal length) partitioning  $[0, 1]$ . Let  $0 = x_0 < x_1 < \dots < x_{n-1} < x_n = 1$ . But for simplicity, we let  $x_i = ih$  for  $i = 0, \dots, n$  where  $h = \frac{1}{n}$ . With this indexing, we get

$$I_{i,h}(f) \equiv \int_{x_{i-1}}^{x_{i+1}} f(x) dx \quad \text{and} \quad T_i(f) = h\{f(x_{i-1}) + f(x_{i+1})\}$$

while  $T_i$  is the basic Trapezoidal rule using  $f(x_{i-1})$  and  $f(x_{i+1})$ . Let  $\overline{T}_i$  be the composite Trapezoidal rule that uses  $f(x_{i-1})$ ,  $f(x_i)$  and  $f(x_{i+1})$ . i.e.,  $\overline{T}_i(f) = \frac{h}{2} \{f(x_{i-1}) + 2f(x_i) + f(x_{i+1})\}$ . Also let  $V_i(f) = \frac{1}{8} (\overline{T}_i(f) - T_i(f))$ .

### 3. An error bound on two consecutive subintervals

In this section, we consider two consecutive subintervals. In order to find a new error bound for the subintervals, we need to compute the distributions of  $V_i$ . In fact, they are Gaussian with zero-mean and are given in next theorem. This is the main result of this paper.

**Theorem.** For  $r \leq 2$ ,

$$M_{\mu_r}(V_i V_j) = \delta_{ij} \cdot c_r \cdot h^{2r+3}, \quad \text{for } i \leq j,$$

where  $\delta_{ij}$  is the Kronecker delta and the constants  $C_r$  are independent of  $h$ 's and equal respectively:  $c_0 = \frac{1}{16}$ ,  $c_1 = \frac{1}{192}$  and  $c_2 = \frac{581}{61440}$ .

**Proof.** Let  $V_{i1} = V_i(f_1)$  and  $V_{i2} = V_i(f_2)$ . Then  $V_i(f) = V_{i1} + V_{i2}$ , and due to the independence of  $f_1$  and  $f_2$ , we have  $M_{\mu_r}(V_i V_j) = M_{\omega_r}(V_{i1} V_{j1}) + M_{\omega_r}(V_{i2} V_{j2})$ . It is easy to see that

$$V_i(f) = \frac{h}{8} \nabla_i f = \nabla_i f_1 + \nabla_i f_2,$$

where  $h = (x_{i+1} - x_{i-1})/2$  and  $\nabla_i f = f(x_{i+1}) - 2f(x_i) + f(x_{i-1})$ . Now,

$$M_{\omega_r}(V_{i1} V_{j1}) = \int_0^1 \left[ -\frac{h}{8} \nabla_i \left( \frac{(\cdot - t)_+^r}{r!} \right) \right] \left[ -\frac{h}{8} \nabla_j \left( \frac{(\cdot - t)_+^r}{r!} \right) \right] dt = \int_0^1 L_{i1}(t) \cdot L_{j1}(t) dt,$$

where  $L_{i1}$  is the first term and  $L_{j1}$  is the second term in above integral. Since  $L_{j1}(t) = 0$  for  $t \leq x_{i+1}$ ,

$$M_{\omega_r}(V_{i1} V_{j1}) = \int_0^{x_{i+1}} L_{i1}(t) \cdot L_{j1}(t) dt.$$

Similarly,

$$M_{\omega_r}(V_{i2} V_{j2}) = \int_{x_{j-1}}^1 L_{i2}(t) \cdot L_{j2}(t) dt,$$

Since we can easily show that the operator  $\nabla_i$  is exact for polynomials of degree  $\leq 2$ ,  $L_{j1} = 0$  for  $t \leq x_{i+1}$  and  $L_{i2} = 0$  for  $x \geq x_j$ . Therefore for  $i < j$ ,  $M_{\mu_r}(V_i V_j) = 0$ .

We now compute the case of  $i = j$  for  $M_{\mu_r}(V_i V_i) = M_{\omega_r}(V_{i1} V_{i1}) + M_{\omega_r}(V_{i2} V_{i2})$ .

$$\begin{aligned} M_{\omega_r}(V_{i1} V_{i1}) &= M_{\omega_r}([V_{i1}]^2) \\ &= \int_{x_{i-1}}^{x_{i+1}} \left[ -\frac{h}{8} \nabla_i \left( \frac{(\cdot - t)_+^r}{r!} \right) \right]^2 dt. \end{aligned} \tag{1}$$

We now compute  $\left[ -\frac{h}{8} \nabla_i \left( \frac{(\cdot - t)_+^r}{r!} \right) \right]^2$  on  $[x_{i-1}, x_{i+1}]$   $[x_{i-1}, x_{i+1}]$ . By the definition of  $\nabla_i$

$$\begin{aligned} &\left[ -\frac{h}{8} \nabla_i \left( \frac{(\cdot - t)_+^r}{r!} \right) \right]^2 \\ &= \left[ -\frac{h}{8} \left( \frac{(x_{i-1} - t)_+^r}{r!} - 2(x_i - t)_+^r r! + \frac{(x_{i+1} - t)_+^r}{r!} \right) \right]^2. \end{aligned} \tag{2}$$

If we apply the equation (1) to (2) by setting  $u = \frac{t - x_{i-1}}{2h}$ , then we have

$$\begin{aligned} &\int_{x_{i-1}}^{x_{i+1}} \left[ -\frac{h}{8} \nabla_i \left( \frac{(\cdot - t)_+^r}{r!} \right) \right]^2 dt \\ &= \int_{x_{i-1}}^{x_{i+1}} \left[ -\frac{h}{8} \left( \frac{(x_{i-1} - t)_+^r}{r!} - 2 \frac{(x_i - t)_+^r}{r!} + \frac{(x_{i+1} - t)_+^r}{r!} \right) \right]^2 dt \\ &= \int_0^1 \left[ -\frac{h}{8} \left( \frac{(2h)^r (0-u)_+^r}{r!} - 2 \frac{(2h)^r (\frac{1}{2} - u)_+^r}{r!} + \frac{(2h)^r (1-u)_+^r}{r!} \right) \right]^2 2h du \\ &= \frac{2^{2r} h^{2r+3}}{32(r!)^2} \int_0^1 \left[ (0-u)_+^r - 2(\frac{1}{2} - u)_+^r + (1-u)_+^r \right]^2 du \\ &= c_{r1} h^{2r+3}, \end{aligned}$$

where

$$c_{r1} = \frac{2^{2r}}{32(r!)^2} \int_0^1 \left[ (0-u)_+^r - 2(\frac{1}{2} - u)_+^r + (1-u)_+^r \right]^2 du.$$

Similarly, we compute  $M_{\omega_r}(V_{i2} V_{i2})$  using  $u = \frac{t - x_{i-1}}{2h}$ .

$$\begin{aligned} M_{\omega_r}(V_{i2} V_{i2}) &= M_{\omega_r}([V_{i2}]^2) \\ &= \int_{x_{i-1}}^{x_{i+1}} \left[ -\frac{h}{8} \nabla_i \left( \frac{(t - \cdot)_+^r}{r!} \right) \right]^2 dt \\ &= \int_{x_{i-1}}^{x_{i+1}} \left[ -\frac{h}{8} \left( \frac{(t - x_{i-1})_+^r}{r!} - 2 \frac{(t - x_i)_+^r}{r!} + \frac{(t - x_{i+1})_+^r}{r!} \right) \right]^2 dt \\ &= \int_0^1 \left[ -\frac{h}{8} \left( \frac{(2h)^r (u-0)_+^r}{r!} - 2(2h)^r (u - \frac{1}{2})_+^r + \frac{(2h)^r (u-1)_+^r}{r!} \right) \right]^2 2h du \\ &= \frac{2^{2r} h^{2r+3}}{32(r!)^2} \int_0^1 \left[ (u-0)_+^r - 2(u - \frac{1}{2})_+^r + (u-1)_+^r \right]^2 du \\ &= c_{r2} h^{2r+3}, \end{aligned}$$

where

$$c_{r2} = \frac{2^{2r}}{32(r!)^2} \int_0^1 \left[ (u-0)_+^r - 2(u - \frac{1}{2})_+^r + (u-1)_+^r \right]^2 du.$$

Since

$$\begin{aligned} &\int_0^1 \left[ (0-u)_+^r - 2(\frac{1}{2} - u)_+^r + (1-u)_+^r \right]^2 du \\ &= \int_0^1 \left[ (u-0)_+^r - 2(u - \frac{1}{2})_+^r + (u-1)_+^r \right]^2 du, \end{aligned}$$

we have  $c_{r1} = c_{r2}$ . Therefore,  $c_r = c_{r1} + c_{r2} = 2c_{r1}$ .

For  $r=0$ ,

$$\begin{aligned} c_0 &= 2c_{01} \\ &= 2 \cdot \frac{1}{32} \int_0^1 \left[ (0-u)_+^0 - 2(\frac{1}{2} - u)_+^0 + (1-u)_+^0 \right]^2 du \\ &= \frac{1}{16} \int_0^{\frac{1}{2}} \left[ (0-u)_+^0 - 2(\frac{1}{2} - u)_+^0 + (1-u)_+^0 \right]^2 du \\ &\quad + \frac{1}{16} \int_{\frac{1}{2}}^1 \left[ (0-u)_+^0 - 2(\frac{1}{2} - u)_+^0 + (1-u)_+^0 \right]^2 du \\ &= \frac{1}{16} \int_0^{\frac{1}{2}} [-2+1]^2 du + \frac{1}{16} \int_{\frac{1}{2}}^1 du \\ &= \frac{1}{16}. \end{aligned}$$

For  $r=1$ ,

$$\begin{aligned}
 c_1 &= 2c_{11} \\
 &= 2 \cdot \frac{1}{32} \int_0^1 \left[ (0-u)_+^1 - 2\left(\frac{1}{2}-u\right)_+^1 + (1-u)_+^1 \right]^2 du \\
 &= \frac{1}{16} \int_0^{\frac{1}{2}} \left[ -2\left(\frac{1}{2}-u\right)_+ + (1-u)_+ \right]^2 du \\
 &\quad + \frac{1}{16} \int_{\frac{1}{2}}^1 \left[ (0-u)_+ - 2\left(\frac{1}{2}-u\right)_+ + (1-u)_+ \right]^2 du \\
 &= \frac{1}{16} \int_0^{\frac{1}{2}} \left[ -2\left(\frac{1}{2}-u\right) + (1-u) \right]^2 du + \frac{1}{16} \int_{\frac{1}{2}}^1 (1-u)^2 du \\
 &= \frac{1}{192} .
 \end{aligned}$$

Finally, for  $r=2$ ,

$$\begin{aligned}
 c_2 &= 2c_{21} \\
 &= 2 \cdot \frac{1}{8} \int_0^1 \left[ (0-u)_+^2 - 2\left(\frac{1}{2}-u\right)_+^2 + (1-u)_+^2 \right]^2 du \\
 &= \frac{1}{4} \int_0^{\frac{1}{2}} \left[ (0-u)_+^2 - 2\left(\frac{1}{2}-u\right)_+^2 + (1-u)_+^2 \right]^2 du \\
 &\quad + \frac{1}{2} \int_{\frac{1}{2}}^1 \left[ (0-u)_+^2 - 2\left(\frac{1}{2}-u\right)_+^2 + (1-u)_+^2 \right]^2 du \\
 &= \frac{1}{4} \int_0^{\frac{1}{2}} \left[ -2\left(\frac{1}{2}-u\right)^2 + (1-u)^2 \right]^2 du \\
 &\quad + \frac{1}{16} \int_{\frac{1}{2}}^1 [(1-u)^2]^2 du \\
 &= \frac{581}{61440} .
 \end{aligned}$$

This completes the proof.

### 3. Conclusion

For Simplicity, we have chosen  $n$  equal length sub-intervals partitioning  $[0, 1]$ . However it is not necessary because if we take  $h$  as the largest length of sub-intervals, then it is straightforward to have the same result. Moreover, if we can compute an error bound for  $r \geq 3$  which is more complicated of course, then it will lead us to compute covariance of average error. We will explore this in the later paper.

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